Duration : 144 minutes

ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Linear Algebra Exam

Common part

Fall 2016

Questions

For the **multiple choice** questions, we give

- +3 points if your answer is correct,
- 0 points if you give no answer or more than one,
- −1 if your answer is incorrect.

The notation and terminology of this exam are those used in the exercise sheets and the lectures of the course Linear Algebra given during the Fall semester 2016.

Notation

– For a matrix A, a_{ij} denotes the entry of A in row i and column j.
– For a vector $\boldsymbol{x}$, x_i denotes the i-the coordinate of $\boldsymbol{x}$.
– I_m denotes the $m \times m$ identity matrix.
– $\mathbb{P}_n$ is the vector space of polynomials of degree less than or equal to n.
– The inner product of vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ is defined as $\boldsymbol{x} \cdot \boldsymbol{y} = \boldsymbol{x}^T \boldsymbol{y}$.
– The length of a vector $\boldsymbol{x} \in \mathbb{R}^n$ is defined as $\|\boldsymbol{x}\| = \sqrt{\boldsymbol{x} \cdot \boldsymbol{x}}$.

Multiple choice questions

For each question, mark the box corresponding to the correct answer. Each question has **exactly one** correct answer.

Question 1 : Consider the matrix

$$A = \begin{pmatrix} 1 & 3 & 3 \\ 3 & 1 & 3 \\ 3 & 3 & 1 \end{pmatrix}.$$

The eigenvalues of A are

- ☐ -2 and 7
- ☐ 3 and 4
- ☐ -5, -1 and 1
- ☐ -2 and 3

Question 2 : Let $T : \mathbb{R}^4 \to \mathbb{R}^2$ be the linear transformation

$$T\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}\right) = \begin{pmatrix} 2x_1 - 3x_2 \\ x_3 + x_1 + x_4 \end{pmatrix}.$$

Then the matrix that represents T in the bases

$$\left\{ \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 2 \end{pmatrix} \right\} \quad \text{and} \quad \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$$

is

- ☐ $\begin{pmatrix} 0 & 1 & 2/3 & 2/3 \\ 1 & -2 & -1/3 & -1/3 \end{pmatrix}$
- ☐ $\begin{pmatrix} 0 & 2 & 7/3 & 2 \\ 2 & -3 & -8/3 & -1 \end{pmatrix}$
- ☐ $\begin{pmatrix} 8 & -2 & 1 & 6 \\ 10 & -7 & -4 & 3 \end{pmatrix}$
- ☐ $\begin{pmatrix} 4 & -4 & -3 & 0 \\ 2 & 1 & 2 & 3 \end{pmatrix}$

Question 3 : Consider the following subsets of $\mathbb{R}^2$:

(a) $\left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$

(b) $\left\{ \begin{pmatrix} a \\ \sin a \end{pmatrix} : a \in \mathbb{R} \right\}$

(c) $\left\{ \begin{pmatrix} 0 \\ a \end{pmatrix} : a \in \mathbb{R} \right\}$

(d) $\left\{ \begin{pmatrix} 0 \\ a^2 \end{pmatrix} : a \in \mathbb{R} \right\}$

(e) $\left\{ \begin{pmatrix} -a/2 \\ -10a \end{pmatrix} : a \in \mathbb{R} \right\}$

Which of these subsets are subspaces of $\mathbb{R}^2$?

- ☐ only (c) and (e)
- ☐ all except (b)
- ☐ all except (d)
- ☐ only (a), (c), and (e)

Question 4 : Let A and B be two $n \times n$ matrices that are similar.
Which of the following statements is not necessarily true?

- ☐ The characteristic polynomials of A and B are the same
- ☐ The ranks of A and B are the same
- ☐ A and B have the same eigenspaces
- ☐ A is diagonalizable if and only if B is diagonalizable

Question 5 : Assume $m \geq 2$. Let A be an $m \times (m-1)$ matrix and let $\boldsymbol{b}$ be a non-zero vector in $\mathbb{R}^m$. Then the set of solutions of $A\boldsymbol{x} = \boldsymbol{b}$ can be

- ☐ the empty set
- ☐ equal to $\mathbb{R}^{m-1}$
- ☐ a subspace of $\mathbb{R}^{m-1}$ of dimension $m-2$
- ☐ a subspace of $\mathbb{R}^{m-1}$ of dimension 1

Question 6 : Let

$$A = \begin{pmatrix} 0 & 0 & -3 \\ 3 & 2 & 0 \\ -1 & \frac{1}{3} & 1 \end{pmatrix}.$$

If $B = A^{-1}$, then the entry b_{12} of B is equal to

- ☐ $-2/3$
- ☐ $1/9$
- ☐ $1/3$
- ☐ $-1/9$

Question 7 : The orthogonal projection of the vector $\begin{pmatrix} 6 \\ 21 \\ 3 \end{pmatrix}$ on the subspace of $\mathbb{R}^3$ generated by $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}$ is

- ☐ $\begin{pmatrix} 2 \\ 16 \\ 11 \end{pmatrix}$
- ☐ $\begin{pmatrix} 10 \\ 26 \\ -5 \end{pmatrix}$
- ☐ $\begin{pmatrix} 4 \\ 8 \\ 7 \end{pmatrix}$
- ☐ $\frac{1}{26}\begin{pmatrix} 255 \\ 396 \\ 375 \end{pmatrix}$

Question 8 : Let $T : \mathbb{R}^4 \to \mathbb{P}_4$ be a linear transformation. Let $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4\}$ be the standard basis of $\mathbb{R}^4$. If the rank of T equals 4, then the set of vectors

$$\{T(\boldsymbol{e}_1 + \boldsymbol{e}_2), T(2\boldsymbol{e}_2), T(\boldsymbol{e}_3 + \boldsymbol{e}_4), T(\boldsymbol{e}_4 + \boldsymbol{e}_1)\}$$

- ☐ cannot be completed to a basis of $\mathbb{P}_4$
- ☐ is not linearly independent
- ☐ can be completed to a basis of $\mathbb{P}_4$
- ☐ is a basis of $\mathbb{P}_4$

Question 9 : Which of the following formulas is always true for two $n \times n$ invertible matrices A and B?

- ☐ $(AB^T)^{-1} = (B^{-1})^T A^{-1}$
- ☐ $(2A)^{-1} = 2^{-n} A^{-1}$
- ☐ $(AB)^{-1} = A^{-1}B^{-1}$
- ☐ $(A + B^T)^{-1} = A^{-1} + (B^{-1})^T$

Question 10 : Let $T : \mathbb{P}_2 \to \mathbb{P}_3$ be the linear transformation $T\big(p(t)\big) = (t+1)p(t)$. Then the matrix that represents T in the bases $\{1, t, t^2\}$ of $\mathbb{P}_2$ and $\{1, t, t^2, t^3\}$ of $\mathbb{P}_3$ is

- ☐ $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$
- ☐ $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$
- ☐ $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$
- ☐ $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

Question 11 : Suppose that $A = \begin{pmatrix} 1 & 0 \\ 3 & 5 \\ 5 & 4 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$. Then the least-squares solution $\widehat{\boldsymbol{x}} = \begin{pmatrix} \widehat{x}_1 \\ \widehat{x}_2 \end{pmatrix}$ of the equation $A\boldsymbol{x} = \boldsymbol{b}$ satisfies

- ☐ $\widehat{x}_2 = 1/6$
- ☐ $\widehat{x}_2 = -35/6$
- ☐ $\widehat{x}_2 = 41/6$
- ☐ $\widehat{x}_2 = -5/6$

Question 12 : Let U be an $n \times p$ matrix with orthonormal columns, and let $W = \mathrm{Col}(U)$. Given a vector $\boldsymbol{y} \in \mathbb{R}^n$, write $\mathrm{proj}_W(\boldsymbol{y})$ for the orthogonal projection of $\boldsymbol{y}$ to W. Then for every vector $\boldsymbol{x} \in \mathbb{R}^p$ and every vector $\boldsymbol{y} \in \mathbb{R}^n$, we have

- ☐ $U^T U\boldsymbol{x} = \mathrm{proj}_W \boldsymbol{x}$ and $UU^T\boldsymbol{y} = \mathrm{proj}_W \boldsymbol{y}$
- ☐ $U^T U\boldsymbol{x} = \boldsymbol{x}$ and $UU^T\boldsymbol{y} = 0$
- ☐ $U^T U\boldsymbol{x} = \boldsymbol{x}$ and $UU^T\boldsymbol{y} = \boldsymbol{y}$
- ☐ $U^T U\boldsymbol{x} = \boldsymbol{x}$ and $UU^T\boldsymbol{y} = \mathrm{proj}_W \boldsymbol{y}$

Question 13 : For which real numbers b does the determinant of the matrix

$$\begin{pmatrix} 2b & 6 & 4 \\ 0 & b-1 & 1 \\ -b & 2b-5 & 5 \end{pmatrix}$$

equal 0?

- ☐ -1 and 1
- ☐ none
- ☐ 0 and -1
- ☐ 0 and 1

Question 14 : Let a, b be two real numbers such that $a + b = 1$, and such that the matrix $A = \begin{pmatrix} 4a & 2 \\ 2 & 4b \end{pmatrix}$ is not invertible. Which of the following statements is true?

- ☐ $\det A = -4$
- ☐ A is a change of basis matrix
- ☐ The characteristic polynomial of A has only one real root
- ☐ The characteristic polynomial of A has two distinct real roots

Question 15 : Let A be a 4×5 matrix such that the matrix equation $A\boldsymbol{x} = \boldsymbol{0}$ has exactly two free variables. What is the dimension of the vector space

$$W = \{\boldsymbol{b} \in \mathbb{R}^4 : A\boldsymbol{x} = \boldsymbol{b} \text{ has a solution}\}?$$

- ☐ 1
- ☐ 0
- ☐ 3
- ☐ 2

Question 16 : let

$$A = \begin{pmatrix} 2 & 4 & 4 \\ 1 & 3 & 1 \\ 1 & 5 & 6 \end{pmatrix}.$$

If $A = LU$ is an LU factorization of A (where L is a lower triangular matrix, whose diagonal entries are equal to 1, and U is an upper triangular matrix), then the the entry ℓ_{32} of L is equal to

- ☐ 3
- ☐ $-3/2$
- ☐ $1/2$
- ☐ $3/2$

Question 17 : Consider the vector space of 3×3 matrices of the form $\begin{pmatrix} 0 & a & 0 \\ b & 0 & c \\ 0 & d & 0 \end{pmatrix}$ with $a, b, c, d \in \mathbb{R}$. Let h be a real parameter. Then the matrices

$$\begin{pmatrix} 0 & 1 & 0 \\ h & 0 & 1 \\ 0 & h & 0 \end{pmatrix}, \begin{pmatrix} 0 & h & 0 \\ 4 & 0 & h \\ 0 & 4 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 3h \\ 0 & 4h & 0 \end{pmatrix}$$

are linearly independent

- ☐ if and only if $h \neq 2$ and $h \neq -2$
- ☐ if and only if $h \neq 2, h \neq -2, h \neq 1/3$, and $h \neq 1/2$
- ☐ for every real value of h
- ☐ if and only if $h \neq 1/2$ and $h \neq 1/3$

Question 18 : Let $A = \begin{pmatrix} -1/2 & 0 & -\sqrt{3}/2 \\ 0 & 1 & 0 \\ \sqrt{3}/2 & 0 & -1/2 \end{pmatrix}$. Which of the following statements are true?

(a) $\det A = 1$ **(b)** $AA^T = I_3$ **(c)** $A^3 = I_3$

- ☐ (a), (b), and (c)
- ☐ only (a) and (c)
- ☐ only (b)
- ☐ only (a) and (b)

Question 19 : The dimension of the subspace of $\mathbb{R}^4$ given by

$$V = \left\{ \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} \in \text{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \\ 1 \end{pmatrix} \right\} \text{ such that } v_4 = 0 \right\}$$

equals

- ☐ 4
- ☐ 1
- ☐ 2
- ☐ 3

Question 20 : Let A and B be two diagonalizable $n \times n$ matrices such that $A \neq B$. Then

- ☐ AB is diagonalizable if A and B have the same eigenvalues
- ☐ AB is never diagonalizable
- ☐ AB is diagonalizable if A and B have the same eigenvectors
- ☐ AB is always diagonalizable

Question 21 : Which of the following statements is true for every $n \times n$ matrix A and every vector $\boldsymbol{b}$ in $\mathbb{R}^n$?

- ☐ The equation $A\boldsymbol{x} = \boldsymbol{b}$ has at most one solution
- ☐ The equation $A\boldsymbol{x} = \boldsymbol{b}$ has at most one least-squares solution
- ☐ The equation $A\boldsymbol{x} = \boldsymbol{b}$ has at least one solution
- ☐ The equation $A\boldsymbol{x} = \boldsymbol{b}$ has at least one least-squares solution

Question 22 : Let

$$A = \begin{pmatrix} 1 & 2 & 1 \\ -3 & -5 & -1 \\ -2 & -4 & -2 \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{b} = \begin{pmatrix} -2 \\ h^3 - h \\ h^3 - 4h + 4 \end{pmatrix},$$

where $h \in \mathbb{R}$ is a parameter. Then the matrix equation

$$A\boldsymbol{x} = \boldsymbol{b}$$

has infinitely many solutions

- ☐ for $h = -2$, $h = 1$, and $h = 2$
- ☐ for $h = -2$, $h = 0$, and $h = 2$
- ☐ for $h = -1$, $h = 0$, and $h = 1$
- ☐ for $h = -1$, $h = -1/2$, and $h = 1/2$

Question 23 : Let b be a real parameter. The polynomial $q(t) = bt - t^2$ belongs to the subspace of $\mathbb{P}_2$ generated by $p_1(t) = 1 + t + t^2$ and $p_2(t) = 2 - t + 3t^2$ for the following value of b:

- ☐ $b = 1$
- ☐ $b = -1$
- ☐ $b = -3$
- ☐ $b = 3$

Question 24 : Let

$$A = \begin{pmatrix} -1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & -1 \\ -2 & -2 & -1 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -1 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ 2 & 3 & 2 & -2 \\ -2 & -2 & -1 & 1 \end{pmatrix}.$$

Then

- ☐ $\dim(\mathrm{Nul}\, A) = 2$ and $\dim(\mathrm{Nul}\, B) = 2$
- ☐ $\dim(\mathrm{Nul}\, A) \neq 2$ and $\dim(\mathrm{Nul}\, B) = 2$
- ☐ $\dim(\mathrm{Nul}\, A) = 2$ and $\dim(\mathrm{Nul}\, B) \neq 2$
- ☐ $\dim(\mathrm{Nul}\, A) \neq 2$ and $\dim(\mathrm{Nul}\, B) \neq 2$